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Dirac observables and spin bases for N relativistic particles

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Abstract. The construction of the Dirac *observables* in the $P^2 > 0$ *stratum* for a system of N relativistic free particles is carried out on the basis of a *quasi*-Shanmugadhasan canonical transformation related to the existence of a Poincaré group action. The explicit form of the Dirac *observables* is derived by exploiting an *internal* Euclidean group having the Poincaré canonical spin as generator of rotations. This procedure provides the *symplectic version* of the conventional angular momentum composition.

1. Introduction

It is generally recognized after Dirac [5] that the Hamiltonian description of the most significant classical systems is based on a pre-symplectic co-isotropically embedded submanifold of the standard phase space, i.e. a manifold equipped with a degenerate *two-form* (in this paper we deal with *first class* constraints only). This entails that the generalized Hamiltonian formalism includes the constraints which define the pre-symplectic submanifold [6, 10, 9, 12, 33, 34, 7]. In particular, the Lagrangians of all the manifestly covariant relativistic systems are *singular* [34] and the *singular* Legendre transformation identifies only a submanifold of the cotangent bundle.

As far as classical physics is concerned, the fundamental issue is the construction of the *observables* for the constrained system (the so-called Dirac *observables*) i.e. the functions on the pre-symplectic manifold that are invariant under the *gauge* transformations generated by the constraints and thereby constant on the *gauge* orbits that foliate the pre-symplectic submanifold itself.

After Shanmugadhasan [31, 19]* a constructive method [19, 20] to find a complete basis of Dirac *observables* has been based on the existence of a specific set of canonical transformations. Within a local chart of the $2M$-dimensional phase space with coordinates (q^i, p_i), covering a region of the pre-symplectic submanifold γ defined by $m < M$ *first-class* constraints $\phi_u(q, p) \approx 0, (u = 1, \ldots, m)$, canonical transformations $(q^i, p_i) \longrightarrow$

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* Let us remark that the well known papers [35] and [36] about field theory, which contain the basic definition of path-integral measure in phase space, are based on the existence of a *guessed* canonical transformation, which was shown by Shanmugadhasan to be actually existent, at least in a finite number of dimensions.

(Q^u, P_u, Q^v, P_v), $(u = 1, \ldots, m; v = 1, \ldots, M - m)$ always exist such that $P_u \approx 0$ define the same region γ. One says that (P_u) is a *local* Abelianization of the *first-class* constraints, that (Q^u) are the associated local Abelian *gauge* variables (locally spanning the intersection of the *gauge* orbits in γ with the given chart) and that (Q^v, P_v) is a *local* complete symplectic basis of 'strong' *Dirac observables* (i.e. not only having zero Poisson brackets with the constraints but also with the Abelian *gauge* variables). Therefore, locally, one gets a separation between *physical* (labelled by v) and *gauge* (labelled by u) degrees of freedom. The local chart (Q^v, P_v, Q^u) realizes a so-called *Darboux chart* on the pre-symplectic submanifold γ. The existence of a Shanmugadhasan canonical transformation for finite-dimensional systems is guaranteed by the Lie theory of function groups (fully developed by Eisenhart [8]).

The main issue concerning classical relativistic systems in general (particles, strings and field configurations) is whether they admit a class of Shanmugadhasan canonical transformations that are defined *globally* over the whole phase space or, at least, in the neighbourhood of the pre-symplectic submanifold γ. If a class of such global transformations exists, and provided one forgets about the issue of manifest Lorentz covariance, one obtains a globally defined separation of the *Dirac observables* from the *gauge* sector of the theory, hence the possibility of reformulating the description of the classical system in terms of *physical* quantities alone.

For many relativistic systems [20] (in particular *particles* [17], *Nambu string* [2–4], *Yang–Mills fields with fermions* [22]) the above procedure is feasible by exploiting the Poincaré group and adapting the variables to its structure. Since this group is always assumed to be globally implementable, its associated momentum map entails precisely the existence of a global Shanmugadhasan canonical transformation.

In this paper we shall apply the Shanmugadhasan technique to the relativistic kinematics of N free scalar particles described by N *first-class* constraints $p_\omega^2 - m_\omega^2 \approx 0$, $\omega = 1 \ldots N$. The constraint pre-symplectic submanifold γ of every relativistic system is foliated in *strata* [1] associated with the four types of Poincaré orbits $P^2 > 0$, $P^2 = 0$, $P^2 < 0$, $P^\mu = 0$. Each *stratum* must be analysed separately since different *little-groups* entail different adaptation of variables. By restricting ourselves to the main *stratum* $(P^2 > 0)$ of the total momentum $P^\mu = \sum_\omega p_\omega^\mu$, and exploiting the semidirect product structure of the Poincaré group, we define a preliminary *centre-of-mass* decomposition (see section 3). The *relative* coordinates, once boosted at rest by the standard Wigner boost technique for the relevant Poincaré orbits, are split into a *rotational scalar* (a *relative time*) and a *Wigner vector*, in agreement with the covariance properties under the $SO(3)$ *little-group* group of the orbits. This is the first step (adaptation of the coordinates to the Poincaré group) towards a Shanmugadhasan canonical transformation. A second step is a nonlinear canonical transformation adapted to $N - 1$ suitable combinations of the N *first-class* constraints. As a matter of fact, this is a *quasi*-Shanmugadhasan canonical transformation, since only $N - 1$ of the new momenta vanish under these constraints. The remaning *first-class* constraint may be put in the form of a polynomial equation of order 2^N in $\sqrt{P^2}$, whose solutions describe the 2^N disjoint branches of the mass spectrum of the system. Therefore 2^{N-1} different Shanmugadhasan canonical transformations adapted to 2^{N-1} branch pairs are required. Note that this final canonical transformation cannot be performed in general if interactions are present (except for the special cases of Liouville integrability. See [21] for a more detailed description of the mass spectrum). At the above stage of the *quasi*-Shanmugadhasan transformation, one of the canonical variables is the square root of the first Poincaré invariant, $\sqrt{P^2}$. For $P^2 > 0$ the second Poincaré invariant is $W^2 = -P^2\boldsymbol{S}^2$ (where $\boldsymbol{S}$ is the rest-frame Thomas canonical spin) and one should take into account the spin–orbits corresponding to both $\boldsymbol{S}^2 \neq 0$ and $\boldsymbol{S}^2 = 0$. In this paper, we shall deal with the main stratum of orbits $\boldsymbol{S}^2 \neq 0$ only, using the constructive theory of the canonical realizations of Lie groups [24–28]. We look for coordinate charts simultaneously adapted to the constraint manifold *and* to the group structure. The adaptation of the coordinates is obtained by means of a homeomorphism between a subset of the *quasi*-Shanmugadhasan chart and the orbits of the co-adjoint action of the Poincaré group (see section 4). In particular, the canonical transformation of the *relative* variables is constructed in such a way that $|\boldsymbol{S}|$ becomes one of the new canonical momenta. In this way, the Darboux chart contains a *typical* chart (in the sense of [24–28]) on the orbits of the co-adjoint representation of the Poincaré group and both of the Poincaré invariants appear in the final canonical basis.

Let us stress that there is a number of motivations for constructing this canonical basis. First of all in the case of special-relativistic systems, the pre-symplectic manifold equipped with the closed degenerated *two-form* must always admit a global pre-symplectic implementation of the Poincaré group. Unfortunately, the studies on pre-symplectic geometry turn out not to be as fully developed as those on other structures: for instance, Poisson manifolds (i.e. the *duals* of pre-symplectic manifolds) have been deeply explored because of their relevance in many sectors of mathematical physics. Therefore, there is a surviving need to develop all the mathematical tools that can be instrumental to the analysis of special-relativistic physical systems with constraints.

As shown in [21], there are several related physical motivations for deepening the understanding of these canonical bases. (i) The need to take into account from the beginning the decomposition of the space of solutions of the physical system in the *strata* corresponding to the various types of Poincaré orbits; note that this decomposition is well known in field theory but it is never really used in applications. (ii) The unsolved problem of relativistic bound states: since the states of Fock space are tensor products of single-particle states, there is no control on the relative times of the *in–out* particles, a fact that is reflected in the spurious solutions of the two-particle Bethe–Salpeter equation. These solutions are excitations of the relative energy, namely the variable conjugated to relative time (incidentally, this was the main motivation for finding the canonical transformation (15)). (iii) The actual impossibility of formulating three (or $N > 3$) *first-class* constraints describing interactions among three (or $N > 3$) scalar particles having the cluster decomposition property in closed form [38, 39] (again a big obstacle in the presence of relative time). Finally: (iv) a related problem is the impossibility of finding the branches of the mass spectrum for $N > 3$ (either free or interacting) particles, since one needs to solve algebraic equations of higher order. A thorough dicussion of these kinematical issues can be found in [21]. There, N free scalar particles are first analysed along the lines of our section 2. Then, this system is reformulated on spacelike hypersurfaces foliating Minkowski spacetime (following Dirac [5]): this reformulation forces one to choose the sign of energy for each particle since the position of a particle on such a hypersurface is identified by three numbers only (all the particles share the same *time* of the hypersurface). Hence, there are 2^N different Lagrangians corresponding to all possible choices. After that, the description is restricted to spacelike hyperplanes and, finally, the configurations belonging to the main stratum $P^2 > 0$ are described on the special family of hyperplanes orthogonal to the total momentum P^μ. Each hyperplane is intrinsically determined by the physical system as its rest frame. The final outcome of this analysis is a new kind of *instant form*, actually the *rest-frame instant form,* characterized by Wigner covariance. This realizes the relativistic separation between the *centre-of-mass* and *relative motions* and allows us to determine the form of the branches of the mass spectrum. In [21] it is also shown that the *equal-time variables* for the N particles (*equal* with respect to the Lorentz-scalar *rest-frame time*) are the same as the subset of relative variables that one obtains in the usual *N-times formulation*, after the canonical transformations leading to our equation (21). Finally, it is shown there that: (i) in the rest-frame hyperplane, a Euclidean kinematical group naturally appears (the Lorentz boosts are implemented as Wigner rotations) and, above all: (ii) action-at-a-distance interactions can be introduced in such a way that the problem of cluster decomposition reduces to the analogous problem in Newtonian mechanics. Furthermore, a system of N charged scalar particles plus the electromagnetic field is described on spacelike hypersurfaces and then reduce to the *rest-frame instant form*.

In conclusion, the mathematical tools developed in this paper can hopefully be exploited in this new *rest-frame instant form* of dynamics in order to investigate unexplored aspects of spin dynamics. As a matter of fact, *our procedure provides the symplectic version of the conventional angular momentum composition*. The method of using a *quasi*-Shanmugadhasan canonical transformation to obtain Dirac *observables* in a *stratum* of the Poincaré group seems to be a powerful and new technique. As we shall see, these new canonical bases may be called '*spin bases*' legitimately, since they describe, at the same time, the *algebraic* and the *geometric* aspects of the composition of classical spins (angular momenta in the *centre-of-mass frame*). In particular, our choice of the Poincaré invariants P^2 and W^2 (i.e. the total invariant mass and angular momentum) as canonical variables (see for example table 5) is the *only choice* of Poincaré invariants that can survive in the interacting case where individual particle Poincaré invariants are no longer constants of the motion. Finally, these '*spin bases*' seem likely to provide an important tool in the study of the Nambu string [2] and in classical field theory.

After some preliminary sections, in section 5, starting from the *quasi*-Shanmugadhasan transformation, we rearrange the *physical relative variables* for a two-particle system in a *typical form* [24] of the canonical realizations of the $E(3)$ group, which turns out to be instrumental for the extension of our construction to systems with a number of particles higher than two. By adapting the chart of the *physical relative variables* $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$ to the $E(3)$ group, we are automatically led to introduce the second Poincaré invariant as a canonical variable and we finally get the main result: a chart that embodies both the Poincaré group structure *and* the constraint structure.

In section 6, starting again from the quasi-Shanmugadhasan transformation, we build up the same construction for a three-particle system. Due to the greater number of degrees of freedom, the procedure is, of course, more cumbersome. The geometry of the *physical relative variables* is characterized by four *Wigner vectors* $\boldsymbol{\pi}_\omega$ and $\boldsymbol{\rho}_\omega$ ($\omega = 1, 2$). By independently transforming the two pairs $(\boldsymbol{\pi}_1, \boldsymbol{\rho}_1)$ and $(\boldsymbol{\pi}_2, \boldsymbol{\rho}_2)$ as in section 5, we obtain a *typical form* of $E(3)$ based on the vector $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$ as generator of the 'internal' rotations. The presence of $\boldsymbol{S}$ as a generator is actually the key factor, since in this way we succeed in adapting the *physical relative variables* to the chosen 'internal' $E(3)$ group, *and*, at the same time, to the Poincaré group, by enrolling the spin invariant as a canonical coordinate. Actually, this method provides the basis for the construction of the '*spin bases*' for a system of N particles. In section 8 the constructive method is first naively applied to a system of N particles starting from the *quasi*-Shanmugadhasan transformation given in section 3, looking for a chart adapted to the Poincaré group. Instructed by the results of section 6, we obtain *chains* of canonical transformations matching a set of canonical subrealizations of the $E(3)$ group, labelled by particle indices. The canonical transformation that leads to the chart adapted to the Poincaré group clearly depends on the combinatorial character of the matching (in some sense, this can be said to be the classical analogue of the spin composition in quantum mechanics). On the other hand, in section 7, it is shown how to invert the canonical transformation that adapts the chart to the Poincaré group and the *geometric* meaning of the *relative canonical variables* is made fully transparent.

2. N-particle constraint theory

As shown by Komar [13–15], a system of N relativistic scalar particles can be described via constraint theory by introducing eight, instead of six, phase-space variables for each particle, namely ($\mu = 0\ldots3$; $\omega = 1\ldots N$):

$$\{x^\mu_\omega, p_{\mu\omega}\} \tag{1}$$

satisfying the usual Poisson algebra

$$\{x^\mu_\omega, p_{\nu\omega'}\} = \delta^\mu_\nu \delta_{\omega\omega'}. \tag{2}$$

In the case of free particles, the N *first-class* constraints ($c = 1$):

$$\chi_\omega \equiv p^2_\omega - m^2_\omega \approx 0 \tag{3}$$

define a pre-symplectic submanifold M_c (co-isotropically embedded in the phase space [10]). The *constraint vector fields* $Y_\omega = \{\cdot\chi_\omega\}$ generate both the dynamics and the *gauge* orbits on M_c since the canonical Hamiltonian is identically zero. The dynamical evolution is ruled by the *Dirac Hamiltonian* $H_D = \sum_\omega \lambda_\omega(\tau)\chi_\omega$ (with $\lambda_\omega(\tau)$ arbitrary multipliers). The Hamilton–Dirac equations imply, for each particle:

$$\begin{aligned} &\boldsymbol{p}_\omega = \text{const} \qquad p^0_\omega = \pm\sqrt{m^2_\omega + \boldsymbol{p}^2_\omega} \\ &x^\mu_\omega(\tau) = x^\mu_\omega(0) + p^\mu_\omega \int_0^\tau \mathrm{d}\tau' \lambda_\omega(\tau'). \end{aligned} \tag{4}$$

These equations provides the manifestly covariant description of the $(\omega)^{\text{th}}$ individual worldline. In order to reduce the remaining seven degrees of freedom for each particle to the six Cauchy data, one has to further reduce the phase space by taking the quotient of M_c with respect to the *gauge* foliation generated by the vector fields Y_ω. On the other hand, within the constraint manifold M_c, the *relative time* among the particles is not restricted and each particle retains seven degrees of freedom, one of which is related to a temporal variable (x^0_ω). The χ_ω's in equations (3) are indeed the generators of the reparametrization invariance transformations on each worldline. From a physical point of view, this means that we have the freedom to define either the 'individual clocks' measuring the time of each particle or a *global time* and $N - 1$ 'relative clocks'. This situation reminds us of the time-diffeomorphism invariance in general relativity where, again, the *gauge* freedom corresponds to the choice of *local clocks*. Finally, the assumed global action of the Poincaré group on M is generated by

$$P^\mu = \sum_\omega p^\mu_\omega \qquad J^{\mu\nu} = \sum_\omega (x^\mu_\omega p^\nu_\omega - x^\nu_\omega p^\mu_\omega) \tag{5}$$

while the Poincaré *invariants* are

$$P^2 \qquad W^2 = -P^2\mathcal{S}^2 \tag{6}$$

where $\mathcal{S}^2 \equiv |\boldsymbol{S}|^2 = -\frac{W^\mu W_\mu}{P^2}$, $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$. M_c is a stratified manifold whose *strata* are identified by the allowed Poincaré orbits defined by P^2, and has $2N$ disjoined components (each particle having its past and future hyperboloids). We shall study the main *stratum* $P^2 > 0$.

3. The Shanmugadhasan transformation

For a single particle with coordinates (x^μ, p_μ) one can perform the following canonical *point* (in p_μ) *transformation* [16] ($i = 1, 2, 3$):

$$\begin{cases} \epsilon = \eta M & T = \eta \frac{p\cdot x}{M} \\ k^i = \eta \dfrac{p^i}{M} & z^i = \eta M \left(x^i - \frac{p^i}{p^0} x^0 \right) \end{cases} \tag{7}$$

where $\eta = \pm$, $M = \sqrt{P^2}$, so that the constraint $\chi \equiv p^2 - m^2 \approx 0$ becomes

$$\chi = \epsilon^2 - m^2 \approx 0. \tag{8}$$

In this case, the manifold M_c has two disjoined components and therefore, setting $\epsilon \longrightarrow \epsilon'_\pm = \epsilon \pm m$, two Shanmugadhasan canonical transformations (SCT) can be defined. The physical meaning of the variables in equations (7) is the following:

- T is the time measured in the rest frame (*centre-of-momentum*) of the particle;
- k^i is the spatial part of the four-velocity $k^\mu = \eta \frac{p^\mu}{M}$ (with $k^0 = (1 + k^2)^{1/2}$ or $k^2 = 1$);
- z^i, modulo a mass factor, are the Cauchy data of the initial *position*.

The canonical pair $T, \epsilon = -1$ describes the *gauge sector*. The constraint (8) fixes the value of ϵ on the physical manifold, while the T variable remains arbitrary. Its *gauge fixing* corresponds to the freedom of arbitrarily choosing the timescale (the *gauge fixing* is equivalent to a definition of the overall clock). The Poincaré generators

$$P^\mu = p^\mu \qquad J^k = \tfrac{1}{2}\epsilon^{ijk} z^i k^j \qquad J^{i0} = k^0 z^i \qquad J^{ij} = z^i k^j - z^j k^i \tag{9}$$

are functions of the variables of the *physical sector* (*observables*). Under the Poincaré transformation (a, Λ), the variables defined in equations (7) transform as (see [16]):

$$\begin{cases} T' = T + k_\mu (\Lambda^{-1} a)^\mu & \epsilon' = \epsilon \\ z'^i = \left(\Lambda^i_{\ j} - \dfrac{\Lambda^i_{\ \mu} k^\mu}{\Lambda^0_{\ \nu} k^\nu} \Lambda^0_{\ j} \right) z^j + \epsilon \left(\Lambda^i_{\ \mu} - \dfrac{\Lambda^i_{\ \nu} k^\nu}{\Lambda^0_{\ \rho} k^\rho} \Lambda^0_{\ \mu} \right) (\Lambda^{-1} a)^\mu \\ k'^\mu = \Lambda^\mu_{\ \nu} k^\nu. \end{cases} \tag{10}$$

Note that z^i are in fact the (non-covariant) Newton–Wigner position variables [23].

For a system of N particles, a SCT must be carried out for each disjoined component of M_c. While this can be easily done for N free particles, the feasibility of the transformation in the presence of an interaction depends upon the Liouville integrability of the interacting system. It is possible, however, to define an *intermediate* SCT (called henceforth *quasi*-Shanmugadhasan canonical transformation) characterized by the fact that the *gauge* sector is separated out, even though the incomplete resolution of the constraints forbids an explicit inversion of the transformation (see [17, 19–21]).

The first step towards the construction of the *quasi*-SCT is the definition of a *preliminary* canonical transformation of the form ($\omega = 1, \ldots, N$; $a = 1, \ldots, N - 1$):

$$\begin{cases} x^\mu = \dfrac{\sum_\omega \hat{\gamma}_\omega x^\mu_\omega}{\sqrt{N}} & P^\mu = \sum_\omega p^\mu_\omega \\ R^\mu_a = \sqrt{N} \displaystyle\sum_\omega \hat{\gamma}_{a\omega} x^\mu_\omega & \Pi^\mu_a = \frac{1}{\sqrt{N}} \sum_\omega \hat{\gamma}_{a\omega} p^\mu_\omega \end{cases} \tag{11}$$

where $\hat{\gamma} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}$, and $\hat{\gamma}_a = (\hat{\gamma}_{a\omega})$ constitute an orthonormal basis of the N-dimensional Euclidean space ($\hat{\gamma}^2 = 1$, $\hat{\gamma}_a^2 = 1$, $\hat{\gamma} \cdot \hat{\gamma}_a = 0$, $\hat{\gamma}_a \cdot \hat{\gamma}_b = \delta_{ab}$, $\hat{\gamma}_\omega \hat{\gamma}_{\omega'} + \sum_a \hat{\gamma}_{a\omega} \hat{\gamma}_{a\omega'} = \delta_{\omega\omega'}$).

The inverse transformation is ($\omega, \omega' = 1, \ldots, N$; $a = 1, \ldots, N-1$):

$$\begin{cases} x_\omega^\mu = x^\mu + \dfrac{\sum_a \hat{\gamma}_{a\omega} R_a^\mu}{\sqrt{N}} \\ p_\omega^\mu = \dfrac{1}{N} P^\mu + \sqrt{N} \displaystyle\sum_a \hat{\gamma}_{a\omega} \Pi_a^\mu \end{cases} \tag{12}$$

so that the constraints of equation (3) can be replaced by

$$\begin{cases} \chi = N \displaystyle\sum_\omega \chi_\omega = P^2 - N \sum_\omega (m_\omega^2 - N \sum_{a,b} \hat{\gamma}_{a\omega} \hat{\gamma}_{b\omega} \Pi_a \cdot \Pi_b) \\ \chi_a = \dfrac{\sqrt{N}}{2} \displaystyle\sum_\omega \hat{\gamma}_{a\omega} \chi_\omega = P \cdot \Pi_a - \frac{\sqrt{N}}{2} \sum_\omega \hat{\gamma}_{a\omega} (m_\omega^2 - N \sum_{b,c} \hat{\gamma}_{b\omega} \hat{\gamma}_{c\omega} \Pi_b \cdot \Pi_c). \end{cases} \tag{13}$$

The Poincaré generators become

$$\begin{aligned} & P^\mu, J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \\ & L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu \\ & S^{\mu\nu} = \sum_a (R_a^\mu \Pi_a^\nu - R_a^\nu \Pi_a^\mu). \end{aligned} \tag{14}$$

A second step, which simplifies the expression of the constraints, consists in performing a further canonical transformation as follows†:

$$\begin{cases} \hat{x}^\mu = x^\mu + \displaystyle\sum_\omega \frac{1}{2P^2} (m_\omega^2 - (p_\omega - \gamma_\omega P) \cdot (p_\omega - \gamma_\omega P)) \\ \quad \times \left(x_\omega^\mu - x^\mu - \left(\displaystyle\sum_{\omega'} \frac{\gamma_{\omega'}^2}{P \cdot p_{\omega'}} \right)^{-1} \sum_{\omega''} \frac{\gamma_{\omega''}^2}{P \cdot p_{\omega''}} \left(\frac{P \cdot r_{\omega\omega''}}{P \cdot p_\omega} (p_\omega^\mu + \gamma_\omega P^\mu) \right. \right. \\ \quad \left. \left. - \displaystyle\sum_{\omega'} \frac{P \cdot r_{\omega'\omega''}}{P \cdot p_{\omega'}} \gamma_{\omega'} p_{\omega'}^\mu \right) \right) \\ P^\mu = \displaystyle\sum_\omega p_\omega^\mu \\ \hat{R}_a^\mu = \sqrt{N} \displaystyle\sum_\omega \hat{\gamma}_{a\omega} \left(x_\omega^\mu + \left(\sum_{\omega'} \frac{\gamma_{\omega'}^2}{P \cdot p_{\omega'}} \right)^{-1} \sum_{\omega''} \frac{\gamma_{\omega''}^2}{P \cdot p_{\omega''}} \frac{P \cdot r_{\omega\omega''}}{P \cdot p_\omega} \cdot (p_\omega^\mu + \gamma_\omega P^\mu) \right) \\ \hat{\Pi}_a^\mu = \dfrac{1}{\sqrt{N}} \displaystyle\sum_\omega \hat{\gamma}_{a\omega} \left(P_\omega^\mu + \frac{P^\mu}{2P^2} \sqrt{N} (m_\omega^2 - (p_\omega - \gamma_\omega P) \cdot (p_\omega - \gamma_\omega P)) \right) \end{cases} \tag{15}$$

where $r_{\omega\omega'}^\mu = x_\omega^\mu - x_{\omega'}^\mu$. Then the $N-1$ constraints χ_a (see equation (13)) can be rewritten in the expressive form:

$$\chi_a = P \cdot \hat{\Pi}_a \approx 0. \tag{16}$$

Note that, for $N > 2$, the transformation (15) cannot be inverted explicitly (see [17, 21]) (for N particles one would have to solve an algebraic equation of order 2^{N-1}). Now, the Poincaré generators are

$$\begin{aligned} & P^\mu, J^{\mu\nu} = \hat{L}^{\mu\nu} + \hat{S}^{\mu\nu} \\ & \hat{L}^{\mu\nu} = \hat{x}^\mu P^\nu - \hat{x}^\nu P^\mu \\ & \hat{S}^{\mu\nu} = \sum_a (\hat{R}_a^\mu \hat{\Pi}_a^\nu - \hat{R}_a^\nu \hat{\Pi}_a^\mu). \end{aligned} \tag{17}$$

† Remember that γ_ω's are equal to $\frac{1}{\sqrt{N}}$.

The final step in the construction of the *quasi*-SCT is achieved in the following way:

(i) first, the *relative* variables are boosted to the *centre-of-mass* rest frame ($\boldsymbol{P} = 0$) ($a = 1, \ldots, N-1$):

$$\begin{cases} r_a^A = L^A_\mu(\overset{0}{P}, P)\hat{R}_a^\mu \\ q_a^A = L^A_\mu(\overset{0}{P}, P)\hat{\Pi}_a^\mu \end{cases} \tag{18}$$

by means of the Wigner boost

$$L^\mu_\nu(P, \overset{0}{P}) = \eta^\mu_\nu + 2\frac{P^\mu \overset{0}{P}_\nu}{P^2} - \frac{(P + \overset{0}{P})^\mu (P + \overset{0}{P})_\nu}{(P + \overset{0}{P}) \cdot P} \qquad \overset{0}{P}{}^\mu \equiv (\eta\sqrt{P^2}, 0, 0, 0) \tag{19}$$

(ii) second, the centre-of-mass coordinates are modified as:

$$\bar{x}^\mu = \hat{x}^\mu - \frac{1}{\epsilon(\epsilon + P_0)}\left(P_\nu \hat{S}^{\nu\mu} + \epsilon\left(\hat{S}^{0\mu} - \hat{S}^{0\nu}\frac{P_\nu P^\mu}{\epsilon^2}\right)\right) \tag{20}$$

and then transformed as in equation (7).

Explicitly, the *quasi*-SCT is given by (see [17]) ($a = 1, \ldots, N-1$):

$$\begin{cases} T = \dfrac{1}{N}\sum_a T_a = \eta\dfrac{P \cdot \bar{x}}{M} \qquad \epsilon = \sum_a \epsilon_a = \eta M \\ z^i = \eta M\left(\bar{x}^i - \dfrac{p^i}{p^0}\bar{x}^0\right) \\ k^i = \eta\dfrac{p^i}{M} \\ \tau_a = \eta\dfrac{P \cdot \hat{R}_a}{M} \qquad \epsilon_a = \eta\dfrac{P \cdot \hat{\Pi}_a}{M} \\ \rho_a^i = \hat{R}_a^i - \dfrac{P^i}{M}\left(\eta\hat{R}_a^0 - \dfrac{\boldsymbol{P} \cdot \hat{R}_a}{M + \eta P_0}\right) \\ \pi_a^i = \hat{\Pi}_a^i - \dfrac{P^i}{M}\left(\eta\hat{\Pi}_a^0 - \dfrac{\boldsymbol{P} \cdot \hat{\Pi}_a}{M + \eta P_0}\right). \end{cases} \tag{21}$$

The variables (21) are reproduced in the *scheme* of table 1: canonically conjugated variables are aligned in the same column. The *gauge* sector lies to the right of the *physical* sector. On the other hand, the *centre-of-mass* variables appear in the upper part and the *relative*

Table 1. *Quasi*-SCT transformation for a system of N particles: chart partially adapted to the constraint structure ($a = 1, \ldots, N-1$).

$\boldsymbol{k}$	$\epsilon = \eta\sqrt{P^2}$
$\boldsymbol{z}$	T
$\boldsymbol{\pi}_a$	ϵ_a
$\boldsymbol{\rho}_a$	τ_a

variables in the lower one. The Poincaré generators can be rewritten in the form

$$\begin{cases} P^\mu = \epsilon k^\mu \\ J^{ij} = z^i k^j - z^j k^i + \sum_a \bar{S}_a^{ij} \qquad \bar{S}_a^{ij} = \rho_a^i \pi_a^j - \rho_a^j \pi_a^i \\ J^{i0} = k^0 z^i + \sum_a \dfrac{\bar{S}_a^{ik} k^k}{1+k_0}. \end{cases} \tag{22}$$

The *quasi*-SCT, summarized in table 1, defines a chart *adapted* to the constraint structure of the theory. The constraints, expressed in the new coordinates, take the form ($a = 1, \ldots, N-1$):

$$\begin{cases} \chi = F(P^2 = \epsilon^2, m_\omega, \boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_b) \qquad \approx 0 \\ \chi_a = \epsilon \epsilon_a \qquad \approx 0 \end{cases} \tag{23}$$

where $F(\cdot)$ is a polynomial of order 2^{N-1} in P^2 whose explicit form can be exhibited only if the canonical transformation (15) can be inverted in a closed form (essentially, one has to solve the system of the $N-1$ equations

$$P \cdot \hat{\Pi}_a = P \cdot \Pi_a - \frac{\sqrt{N}}{2} \sum_\omega \hat{\gamma}_{a\omega} \left(m_\omega^2 - N \sum_{b,c} \hat{\gamma}_{b\omega} \hat{\gamma}_{c\omega} \left(\frac{P \cdot \Pi_b P \cdot \Pi_c}{P^2} - \hat{\boldsymbol{\Pi}}_b \cdot \hat{\boldsymbol{\Pi}}_c \right) \right)$$
$$(a = 1, \ldots, N-1)$$

to obtain $P \cdot \Pi_a$ in terms of $P \cdot \hat{\Pi}_a$ and of $\hat{\boldsymbol{\Pi}}_b \cdot \hat{\boldsymbol{\Pi}}_c$. Here $\hat{\boldsymbol{\Pi}}_a$ is the space part of Π_a^μ defined in equation (11), taken after having boosted it to the *centre-of-mass* frame ($\boldsymbol{P} = 0$)). Note, in addition, that the Wigner boost splits the *relative* variables in *rotational scalars* τ_a, ϵ_a and in *Wigner vectors* $\boldsymbol{\rho}_a$, $\boldsymbol{\pi}_a$, (see [21] for a more detailed discussion of this kinematical setting; the variables $\hat{x}^\mu$, $\hat{R}_a^\mu$, $\hat{\Pi}_a^\mu$, $\bar{x}^\mu$, T, $\boldsymbol{\rho}_a$, τ_a, $\boldsymbol{\pi}_a$, ϵ_a of this paper are denoted in [21] by $\hat{x}^\mu$, $\hat{R}_a^\mu$, $\hat{Q}_a^\mu$, $\hat{\tilde{x}}^\mu$, $\hat{T}$, $\hat{\boldsymbol{\rho}}_a$, $\hat{T}_{Ra}$, $\hat{\boldsymbol{\pi}}_a$, $\hat{\epsilon}_{Ra}$, respectively).

Since the evolution in an arbitrary parameter τ is ruled by the Dirac Hamiltonian $H_D = \lambda(\tau)\chi + \sum_a \lambda_a(\tau)\chi_a$, it is seen that only $\bar{x}^\mu$ has its four-velocity parallel to the total four-momentum: $\frac{\mathrm{d}}{\mathrm{d}\tau}\bar{x}^\mu = \{\bar{x}^\mu, H_D\} = 2\lambda(\tau)\frac{\partial F}{\partial P^2} P^\mu$. On the other hand, from $\frac{\mathrm{d}}{\mathrm{d}\tau} x^\mu = \lambda(\tau)\{x^\mu, \chi\} + \sum_a \lambda_a(\tau)\{x^\mu, \chi_a\}$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{x}^\mu = \lambda(\tau)\{\hat{x}^\mu, \chi\} + \sum_a \lambda_a(\tau)\{\hat{x}^\mu, \chi_a\}$, it follows that x^μ and $\hat{x}^\mu$ exhibit a motion with respect to $\bar{x}^\mu$, which we shall denominate *pseudo-Zitterbewegung*. In general, this phenomenon naturally arises from the conjunction of the requirements that a 'position' dynamical variable be at the same time canonical and relativistically covariant. As is well known, only three different 'position' dynamical variables can be defined within the *irreducible* realizations of the Poincaré group (namely the Moeller, the Fokker and the Pryce–Newton–Wigner three-vector *centres of mass*) but none of these is at the same time canonical and covariant. If, however, one is not restricted to Poincarè algebra generators alone and extra degrees of freedom are allowed, an indefinite number of 'position' dynamical variables can be constructed (see the relevant references in [29, 30]). In particular, a Dirac-like canonical and covariant three-vector position variable $\boldsymbol{x}(t)$ can be constructed (in the *instant form*) in the case of the *reduced* two-particle model (*pole–dipole* of [29, 30]) in terms of the Poincaré generators and of an additional *inner* angular degree of freedom canonically conjugated to the magnitude of the intrinsic angular momentum. The time derivative of this vector is not parallel to the total momentum so that it undergoes an helicoidal motion which can safely be called a classical *Zitterbewegung*. On the other hand, in the context of this paper, the extra degrees of freedom belong to the *gauge* sector, so that the canonical and covariant position vector $\hat{\boldsymbol{x}}$ (see section 3) is, necessarily, only partially adapted to the constraints. Relative to the Fokker three-vector $\boldsymbol{X}_F = -\frac{\boldsymbol{K}}{P_0} + \frac{\bar{\boldsymbol{S}}\wedge\boldsymbol{P}}{\epsilon P_0}$, interpreted as a *geometric* (covariant but non-canonical) *centre of mass*, $\hat{\boldsymbol{x}}$ undergoes a motion that can be seen as 'internal' with respect to the particle system as a whole. The terminology of *pseudo-Zitterbewegung* in this case appears to be justified by the usage already established in the literature† and by the fact that $|\hat{\boldsymbol{x}} - \boldsymbol{X}_F|$ has a typical relation with the invariants of the 'spin' Lorentz group (precisely, in the case of two particles, in the rest frame $\boldsymbol{P} = 0$, one has‡ $|\hat{\boldsymbol{x}} - \boldsymbol{X}_F|_{\boldsymbol{P}=0} = \frac{1}{\epsilon}(\bar{T}_R^2\pi^2 + \bar{\epsilon}_R^2[(\frac{\rho\cdot\pi}{\pi})^2 + \frac{S^2}{\pi^2} + 2\bar{T}_R\bar{\epsilon}_R\frac{\rho\cdot\pi}{\pi}\pi)^{1/2} = \frac{\sqrt{S^2-I_1}}{\epsilon}$, where I_1 is the Lorentz invariant $\frac{1}{2}S^{\mu\nu}S_{\mu\nu}$). Finally, it is obvious by construction that all the position variables we take into account, in the non-relativistic limit and when evaluated at the same time, reduced to the standard Newtonian centre of mass. These issues will be exhaustively discussed elsewhere.

Under the *quasi*-SCT, the *centre-of-mass* coordinates have also been automatically adapted to the co-adjoint orbits of the Poincaré group and the invariant P^2 now appears among the canonical variables. However, the second invariant of the Poincaré group, namely the Pauli–Lubanski invariant (see (9)),

$$W^2 = -P^2\boldsymbol{S}^2 \qquad (S^i = \epsilon^{ijk}\sum_a \bar{S}_a^{jk}) \tag{24}$$

is not enrolled among the *relative* canonical variables and the canonical realization of the Poincaré group has not assumed, as yet, a *typical form*§. The completion of our programme demands adapting all of the coordinates to the co-adjoint action of the Poincaré group: this requires in turn that the *relative* variables be adapted to the $SO(3)$ group, which is the small group of the orbits we are analysing.

The basic tool for realizing this step is provided by the constructive theory of the canonical realizations of Lie groups sketched in the following section. It will appear that exploiting the geometrical *and* group properties of the *relative physical* variables needs the implementation of a *second-rank* group containing $SO(3)$ as a proper subgroup, such as $E(3)$, $SO(3,1)$ or $SO(4)$.

4. Canonical realizations of a Lie group

Let M be a symplectic manifold and Φ the action of the Lie group G on M:

$$\Phi : G \times M \to M. \tag{25}$$

It can be shown that it is possible to construct a peculiar class of local charts on M by exploiting the local homeomorphism existing between a submanifold of M and the co-adjoint orbits on $\mathcal{G}^*$ (dual of the Lie algebra $\mathcal{G}$ of G) [1, 24, 32]. Each chart belonging to the above class, characterized by it being adapted to the group structure, is called *typical form*.

According to the general theory [24], a canonical realization $\mathcal{K}$ of a Lie group G_r (of order r) can be characterized in terms of two basic *schemes*: (A) the *scheme A* which depends entirely on the structure of the Lie algebra $\mathcal{L}_{G_r}$ (including its *cohomology*) and amounts to a *pseudo-canonization* of the generators, in terms of k *invariants* $\mathcal{J}_1 \ldots \mathcal{J}_k$ and $h = (r-k)/2$ pairs of canonical variables (*irreducible kernel* of the *scheme*), functions

† See for instance the terminology adopted in [37] in connection with the position operator for the field configurations in the Klein–Gordon case.

‡ See section 5 for the definition of the variables.

§ See section 4.

of the generators; (B) the *scheme B* (or *typical form*) which is an array of $2n$ canonical variables P_i, Q_i, defined by means of a *canonical completion* of the *scheme A*. Locally, *scheme B* allows us to *analyse* any *given* canonical realization of G_r and to *construct* the most general canonical realization of G_r.

A generic *scheme A* can be usefully visualized by the following table:

$$\begin{array}{|ccc|ccc|} \hline \mathcal{P}_1 & \dots & \mathcal{P}_h & \mathcal{J}_1 & \dots & \mathcal{J}_k \\ \mathcal{Q}_1 & \dots & \mathcal{Q}_h & & & \\ \hline \end{array} \tag{26}$$

where variables belonging to the same vertical pair are canonical conjugated, and variables belonging to different vertical lines commute. The quantities $\mathcal{J}_i$ clearly commute with all of the generators and are the *invariants* of the realization. Of course, any set of k functional independent functions $\mathcal{J}'_1(\mathcal{J}_1, \dots, \mathcal{J}_k), \dots, \mathcal{J}'_k(\mathcal{J}_1, \dots, \mathcal{J}_k)$ of the *invariants* are good *invariants* as well. A *scheme A* is called *singular* if, due to some functional relations, already existing or imposed, among the *invariants*, some canonical pairs become singular functions of the generators and must be omitted from *scheme A* itself which, then, has to be redetermined from the beginning. In this case, the number of *canonical pairs* may result in $m < h$.

The *scheme B*, i.e. the *canonical completion* of *scheme A*, is accomplished in general according to one of the following possibilities.

Type 1. No new variable is added to *scheme A* and k independent functions of the *canonical invariants* are put identically equal to constants. $2h$ *typical* variables P_i, Q_j are identified with the variables of the *irreducible kernel*. The explicit expression of the canonical generators, in terms of the P_i, Q_j of *scheme B*, is obtained by inverting the functions of *scheme A*. Then, an arbitrary *fixed* (with respect to the group parameters) canonical transformation $\mathcal{S}$ gives the expression of the generators in terms of $2h$ generic canonical variable p_i, q_j. Since, in this case, the phase space contains no submanifolds invariant under the action of G_r, these realizations are called *irreducible* (i.e. *transitive*), for any k*-tuple* of allowed constant values of the functions of the invariants.

Type 2. A certain number l ($l \leqslant k$) of canonical variables Q_s (called *supplementary* variables) turn out to be coupled or are axiomatically coupled, to l of the invariants $\mathcal{J}_i$, building up l new canonical pairs, while $k - l$ independent functions of the *invariants* are put identically equal to constants. Then, the generators, as functions of the *typical* variables $P_i, Q_i, P_s (i, j = 1, \dots, h; s = 1, \dots, l)$, are obtained by inversion as before but do not depend on the *supplementary* variables Q_s. The phase space, in this case, is $2h + 2l$ dimensional and, containing the submanifolds $\mathcal{J}_t(p, q) =$ constant $(t = 1, \dots, k - l)$ as invariant submanifolds, corresponds to *non-irreducible* (i.e. *intransitive*) realizations for any $(k-l)$*-tuple* of allowed constant values of the invariant functions that have been constrained. A particular case of type 2 is the so-called *complete* realization, corresponding to $l = k$ and $(r + k)/2$ canonical pairs. This realization is completely determined, locally, by the group structure. In geometrical terms, the variables of the *irreducible kernel*, together with the k *supplementary* variables, define a local chart on the orbits of the co-adjoint representation of G_r.

Type 3. An arbitrary number $v \leqslant n - h$ of pairs of canonical variables Q_u, $P_u (u = 1, \dots, v)$ (*inessential* variables) turn out to exist or are axiomatically added to *scheme A*. These variables are not canonically coupled to *invariants* nor do they share any functional relation with the variables of *scheme A*, so that they commute with all the variables considered up to now. Then, one proceeds, as for type 1 and type 2, by inverting the functional dependence of the *typical* variables on the generators and performing an arbitrary *fixed* canonical transformation which leads to the generic form of the realization in terms of $2n$ canonical variables p_i, q_j. Since the *inessential* variables define invariant submanifolds in phase-space, these realizations are *non-irreducible* (i.e. *intransitive*). Note that types 1-and-3, or 2-and-3, are mutually compatible.

A priori, we are interested in the rotation group $SO(3)$, the Poincaré group, and the rank-2 groups $E(3)$, $SO(3,1)$ and $SO(4)$. Their *schemes A* have the following structure.

SO(3). The Poisson bracket algebra (PBA) and the *scheme A* are

$$\{S^i, S^j\} = \epsilon^{ijk} S^k \tag{27}$$

and

$$\boxed{\begin{array}{l|l} \mathcal{P}_1 = S^3 & \mathcal{J}_1 = \sqrt{(S^1)^2 + (S^2)^2 + (S^3)^2} \equiv \mathcal{S} \\ \mathcal{Q}_1 = \arctan \dfrac{S^2}{S^1} & \end{array}} \tag{28}$$

respectively.

$T^4 \widehat{\times} SO(3,1)$ *(Poincaré group).* The generators are: $\boldsymbol{J}$ rotations, $\boldsymbol{K}$ special Lorentz transformations, $P^\mu = (P_0, \boldsymbol{P})$ spacetime translations. The PBA is

$$\begin{array}{ll} \{J_i, J_j\} = \epsilon_{ij}{}^k J_k & \{K_i, K_j\} = -\epsilon_{ij}{}^k J_k \\ \{J_i, K_j\} = \epsilon_{ij}{}^k K_k & \{K_i, P_j\} = -\delta_{ij} P_0 \\ \{J_i, P_j\} = \epsilon_{ij}{}^k P_k & \{K_i, P_0\} = -P_i \\ \{J_i, P_0\} = 0 & \{P_i, P_0\} = 0 \\ \{P_i, P_j\} = 0 & \end{array} \tag{29}$$

and the *scheme A* for the $P^2 > 0$ class can be written as

$$\boxed{\begin{array}{ll|l} \boldsymbol{\mathcal{P}} = \boldsymbol{P} & \mathcal{P}_4 = S^3 & \mathcal{J}_1 = \mathcal{S} \quad \mathcal{J}_2 = M \\ \boldsymbol{\mathcal{Q}} = -\dfrac{\boldsymbol{K}}{P_0} + \dfrac{\boldsymbol{S} \wedge \boldsymbol{P}}{P_0(P_0 + \sqrt{\Pi})} & \mathcal{Q}_4 = \arctan \dfrac{S^2}{S^1} & \end{array}} \tag{30}$$

where

$$\begin{cases} M^2 = P_0^2 - \boldsymbol{P}^2 \\ S^i = \dfrac{P_0}{\sqrt{\Pi}} J^i + \dfrac{(\boldsymbol{K} \wedge \boldsymbol{P})^i}{\sqrt{\Pi}} - \dfrac{\boldsymbol{J} \cdot \boldsymbol{P}}{\sqrt{\Pi}(P_0 + \sqrt{\pi})} P^i \\ \mathcal{S} = \sqrt{(S^1)^2 + (S^2)^2 + (S^3)^2}. \end{cases} \tag{31}$$

The *rank-2* groups $E(3)$, $SO(3,1)$ and $SO(4)$ have the following PBAs:

E(3).

$$\{S^i, S^j\} = \epsilon^{ijk} S^k \qquad \{S^i, R^j\} = \epsilon^{ijk} R^k \qquad \{V^i, R^j\} = 0 \tag{32}$$

SO(3,1).

$$\{S^i, S^j\} = \epsilon^{ijk} S^k \qquad \{S^i, R^j\} = \epsilon^{ijk} R^k \qquad \{R^i, R^j\} = -\epsilon^{ijk} S^k \tag{33}$$

SO(4).

$$\{S^i, S^j\} = \epsilon^{ijk} S^k \qquad \{S^i, R^j\} = \epsilon^{ijk} R^k \qquad \{R^i, R^j\} = \epsilon^{ijk} S^k \tag{34}$$

while the corresponding *schemes* A can be given the unique form

S^3	$\mathcal{S}$	I_1	I_2
$\tan^{-1}\frac{S^2}{S^1}$	$\alpha = \tan^{-1}\frac{\mathcal{S}(\boldsymbol{S}\wedge\boldsymbol{R})^3}{[\boldsymbol{S}\wedge(\boldsymbol{S}\wedge\boldsymbol{R})]^3}$		

(35)

provided the *invariants* are defined as

$$\begin{aligned} &I_1 = \boldsymbol{R}\cdot\boldsymbol{R} && I_2 = \boldsymbol{R}\cdot\boldsymbol{S} \\ &I_1 = \boldsymbol{S}^2 - \boldsymbol{R}^2 && I_2 = \boldsymbol{R}\cdot\boldsymbol{S} \\ &I_1 = \boldsymbol{S}^2 + \boldsymbol{R}^2 && I_2 = \boldsymbol{R}\cdot\boldsymbol{S} \end{aligned} \tag{36}$$

respectively, in the three cases.

5. System of two particles

The simplest model in which both the constraints *and* the canonical group structure can be significantly exhibited is provided by a system of two particles. Table 2 summarizes the *quasi*-SCT in this case.

As anticipated at the end of section 3, the next step in the construction of the final basis consists of adapting the *relative physical* variables $(\boldsymbol{\rho}, \boldsymbol{\pi})$ not only to the small $SO(3)$ group, but rather to a larger *second-rank* group. In what follows we shall exploit the Euclidean group $E(3)$ since its Abelian invariant subgroup is particularly suited to be realized in terms of the 'internal' *relative* variables, and the role of the Abelian subgroup is a key factor to the whole construction.

First of all, the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\pi}$ are replaced by the scalar *radial* variables†

$$\pi \equiv \sqrt{\boldsymbol{\pi}^2} \qquad \boldsymbol{\rho}\cdot\hat{\pi} \tag{37}$$

and the new vectors $(i, j = 1, 2, 3)$

$$\hat{\pi}^i \equiv \frac{\pi^i}{\pi} \qquad \xi^i \equiv \pi\rho^i_\perp \equiv \sqrt{\boldsymbol{\pi}^2}(\delta^{ij} - \hat{\pi}^i\hat{\pi}^j)\rho^j \qquad (\hat{\pi}\cdot\boldsymbol{\rho}_\perp = 0) \tag{38}$$

so that the canonical spin can be written:

$$\boldsymbol{S} = \boldsymbol{\rho}\wedge\boldsymbol{\pi} = \sqrt{\boldsymbol{\pi}^2}\boldsymbol{\rho}_\perp\wedge\hat{\pi} = \boldsymbol{\xi}\wedge\hat{\pi}. \tag{39}$$

Then, we obtain an explicit expression of the *irreducible kernel* of the *scheme* (35) by realizing the vector $\boldsymbol{R}$ simply as

$$\hat{R}^i = \hat{\pi}^i. \tag{40}$$

Since $\{\hat{\pi}^i, \hat{\pi}^j\} = 0$, $\boldsymbol{S}\cdot\hat{\pi} = 0$ and $\hat{\pi}\cdot\hat{\pi} = 1$, this automatically implements a realization of the Euclidean group $E(3)$, characterized by the fixed values $I_1 = \boldsymbol{R}\cdot\boldsymbol{R} = 1$, $I_2 = \boldsymbol{R}\cdot\boldsymbol{S} = 0$

† For any vector $\boldsymbol{V}$, the expression $\hat{V}$ means $\frac{\boldsymbol{V}}{V}$ throughout the paper.

Table 2. *Quasi*-SCT for a system of two particles: chart partially adapted to the constraint structure.

$\boldsymbol{k}$	$\epsilon = \eta\sqrt{P^2}$
$\boldsymbol{z}$	T
$\boldsymbol{\pi}$	ϵ_R
$\boldsymbol{\rho}$	τ_R

of the invariants, and by a functional form of the variable canonically conjugated to $\mathcal{S}$ given by†:

$$\alpha = \tan^{-1}\frac{1}{\mathcal{S}}(\boldsymbol{\pi}^2\frac{\rho^3}{\pi^3} - \boldsymbol{\rho}\cdot\boldsymbol{\pi}) = \tan^{-1}\frac{1}{\mathcal{S}}\frac{\xi^3}{\hat{\pi}^3}. \tag{41}$$

Note that the following formulae hold true:

$$\frac{\partial\hat{R}(\alpha)}{\partial\alpha} = \hat{R}(\alpha + \frac{\pi}{2}) = \frac{\boldsymbol{S}}{\mathcal{S}}\wedge\hat{R}(\alpha) \tag{42}$$

and

$$\begin{cases} \hat{R}^1 = \dfrac{S^2}{\sqrt{\mathcal{S}^2 - (S^3)^2}}\sin\alpha - \dfrac{S^1S^3}{\mathcal{S}\sqrt{\mathcal{S}^2 - (S^3)^2}}\cos\alpha \\ \hat{R}^2 = \dfrac{S^1}{\sqrt{\mathcal{S}^2 - (S^3)^2}}\sin\alpha - \dfrac{S^2S^3}{\mathcal{S}\sqrt{\mathcal{S}^2 - (S^3)^2}}\cos\alpha \\ \hat{R}^3 = \dfrac{\sqrt{\mathcal{S}^2 - (S^3)^2}}{\mathcal{S}}\cos\alpha \end{cases} \tag{43}$$

so that the inverse transformation of the *relative* variables can be written:

$$\begin{cases} \pi^i = \pi\hat{R}^i(\boldsymbol{S},\alpha) \\ \rho^i = \dfrac{\boldsymbol{\pi}\cdot\boldsymbol{\rho}}{\pi}\hat{R}^i(\boldsymbol{S},\alpha) + \dfrac{\mathcal{S}}{\pi}\dfrac{(\hat{R}(\boldsymbol{S},\alpha)\wedge\boldsymbol{S})^i}{\mathcal{S}}. \end{cases} \tag{44}$$

Finally, owing to the Poisson bracket relations:

$$\{S^i,\pi\} = \{S^i,\rho\cdot\hat{\pi}\} = 0 \qquad \{\alpha,\pi\} = \{\alpha,\rho\cdot\hat{\pi}\} = 0 \tag{45}$$

and

$$\{\rho\cdot\hat{\pi},\pi\} = 1 \tag{46}$$

it follows that the $E(3)$ canonical realization has the *radial* variables as *inessential* variables and therefore is a *non-irreducible* canonical realization of type 3 (see section 4), summarized in table 3. Since now the Poincaré invariant $\mathcal{S}$ has been enrolled as a canonical variable, the adaptation to the Poincaré group is complete. Table 4 summarizes the *typical form* of the Poincaré realization, displaying the same distinctions made before between *gauge* and *physical* variables as well as between *centre-of-mass* and *relative* variables.

† Of course the superscripts 3 in the following expressions means 'third component of a three-vector'. Its geometric meaning is shown in figure 1.

Table 3. System of two particles: *relative physical* variables adapted to the internal $E(3)$ group.

S^3	$\mathcal{S}$	$R^2 = 1 \quad \boldsymbol{R}\cdot\boldsymbol{S} = 0$	π
$\tan^{1-}\frac{S^2}{S^1}$	α		$\frac{\pi\cdot\rho}{\pi}$

Table 4. System of two particles: chart adapted to the constraint structure *and* to the Poincaré group.

k^1	k^2	k^3	$\epsilon = \eta\sqrt{P^2}$
z^1	z^2	z^3	T
S^3	$\mathcal{S}$	π	ϵ_R
$\tan^{-1}\frac{S^2}{S^1}$	α	$\frac{\pi\cdot\rho}{\pi}$	τ_R

Note that the vectors $\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{S}\wedge\boldsymbol{R}$ define a right-handed orthogonal reference frame (see figure 1) and satisfy the following peculiar algebraic relations:

$$
\begin{aligned}
&\{S^i, S^j\} = \epsilon^{ijk}S^k \qquad \{R^i, R^j\} = 0 \qquad \{S^i, R^j\} = \epsilon^{ijk}R^k \\
&\{(\boldsymbol{S}\wedge\boldsymbol{R})^i, S^j\} = \epsilon^{ijk}(\boldsymbol{S}\wedge\boldsymbol{R})^k \\
&\{(\boldsymbol{S}\wedge\boldsymbol{R})^i, (\boldsymbol{S}\wedge\boldsymbol{R})^j\} = -R^2\epsilon^{ijk}S^k \\
&\{(\boldsymbol{S}\wedge\boldsymbol{R})^i, R^j\} = R^iR^j - R^2\delta^{ij}.
\end{aligned}
\tag{47}
$$

6. The system of three particles as a canonical realization of the Euclidean group $\boldsymbol{E(3)}$

The simplest model in which both the constraint *and* the Poincaré group structures are substantially exploited, and the Euclidean *second-rank* group plays an effective instrumental role, is provided by a system of three particles. The initial scheme is given in table 1. According to our programme, we have to adapt the *relative physical* variables (ρ^i_a, π^j_a), $(a = 1, 2)$ to the $E(3)$ co-adjoint orbits. Since variables with different particle labels commute, table 1 can be rewritten as the 'direct product' of two $E(3)$ realizations constructed as in section 5. In fact, we have now a *pair* of commuting reference frames defined by the two sets of $E(3)$ generators, namely:

$$\hat{R}_1, \boldsymbol{S}_1, \hat{S}_1\wedge\boldsymbol{R}_1 \qquad \text{and} \qquad \hat{R}_2, \boldsymbol{S}_2, \hat{S}_2\wedge\boldsymbol{R}_2 \tag{48}$$

the canonical spin being

$$\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2. \tag{49}$$

The two *irreducible kernel* of the $E(3)$ realizations are

S_1^3	$\mathcal{S}_1$	S_2^3	$\mathcal{S}_2$
$\tan^{-1}\frac{S_1^2}{S_1^1}$	α_1	$\tan^{-1}\frac{S_2^2}{S_2^1}$	α_2

. (50)

We will construct now, starting from the two previous single-particle realizations of $E(3)$, a global canonical realization of a *single* $E(3)$ with $\boldsymbol{S}$ given by (49) and with fixed values of the invariants (as in section 5). Defining a unit vector $\hat{R}$ such that:

$$\hat{R}^2 = 1 \qquad \hat{R}\cdot\boldsymbol{S} = 0 \qquad \{\hat{R}^i, \hat{R}^j\} = 0 \tag{51}$$

and making the *ansatz*

$$\hat{R} = a\hat{R}_1 + b\hat{R}_2 \tag{52}$$

equations (51) are satisfied by the following values of a and b:

$$\frac{a}{b} = -\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2} \qquad b = \left(1 + \left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)^2 - 2(\hat{R}_1 \cdot \hat{R}_2)\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)\right). \tag{53}$$

Then, the *irreducible kernel* of the $E(3)$ realization in terms of the *relative physical* variables π_a^i, ρ_a^j is still given by the *scheme* of equation (35), if $\boldsymbol{R}$ is replaced by $\hat{R}$ (see equation (52)). This naturally defines the normalized *common* reference frame:

$$\hat{S} \equiv \frac{\boldsymbol{S}}{\mathcal{S}} \qquad \hat{R} \qquad \hat{\xi} \equiv \hat{S} \wedge \hat{R}. \tag{54}$$

A canonical completion of the *typical form* requires the construction of *four* new *inessential* variables which, together with the single-particle *radial* variables $(\pi_a, \hat{\pi}_a \cdot \boldsymbol{\rho}_a)$, build up the right number of deegres of freedom. The new variables have to be guessed by exploiting the geometry of the reference frames (48) whose *relative orientation* is not constrained: actually, the *inessential* variables to be constructed do supply precisely the geometrical information expressing the arbitrary relative orientation.

First of all, the following scalar functions can be shown to commute with the angle α, defined in *scheme A* of equation (35):

$$\begin{gathered}
\hat{R}_1 \cdot \hat{R} \qquad \hat{R}_1 \cdot \hat{\xi} \qquad \hat{R}_1 \cdot \hat{\xi}_2 \\
\hat{R}_2 \cdot \hat{R} \qquad \hat{R}_2 \cdot \hat{\xi} \qquad \hat{R}_2 \cdot \hat{\xi}_1 \\
\frac{\hat{R}_1 \cdot \boldsymbol{S}}{1 - (\hat{R}_1 \cdot \hat{R}_2)^2} + \frac{1}{2}\mathcal{S} \qquad \frac{\hat{R}_2 \cdot \boldsymbol{S}}{\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)((\hat{R}_1 \cdot \hat{R}_2)^2) - 1} - \frac{1}{2}\mathcal{S} \\
\frac{\boldsymbol{S}_1 \cdot \hat{\xi}}{\hat{R}_1 \cdot \boldsymbol{S}} \qquad \frac{\boldsymbol{S}_2 \cdot \hat{\xi}}{\hat{R}_1 \cdot \boldsymbol{S}} \\
\boldsymbol{S}_1 \cdot \boldsymbol{S} + \frac{1}{2}\mathcal{S}^2\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)^2 \qquad \boldsymbol{S}_2 \cdot \boldsymbol{S} + \frac{1}{2}\mathcal{S}^2\left(1 - \left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)^2\right).
\end{gathered} \tag{55}$$

Then, a possible choice of the remaining *inessential* canonical variables, respecting the particle-label symmetry, is the following:

$$\begin{aligned}
&p_1 = \hat{R}_1 \cdot \boldsymbol{R} \qquad p_2 = \hat{R}_2 \cdot \boldsymbol{R} \\
&q^1 = A_1(p_1, p_2)\frac{(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\hat{R}_1 \wedge \hat{R}_2)}{2(1 - (\hat{R}_1 \cdot \hat{R}_2)^2)} - B_1(p_1, p_2)\frac{\hat{R}_1 \cdot \hat{\xi}}{\mathcal{R}^2} \\
&q^2 = A_2(p_1, p_2)\frac{(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\hat{R}_1 \wedge \hat{R}_2)}{2(1 - (\hat{R}_1 \cdot \hat{R}_2)^2)} - B_2(p_1, p_2)\frac{\hat{R}_1 \cdot \hat{\xi}}{\mathcal{R}^2}
\end{aligned} \tag{56}$$

where

$$
\begin{aligned}
A_1 &= -\tfrac{1}{2}(\hat{R}_1 \cdot \hat{R}_2) A_2 \\
B_1 &= -(\hat{R}_1 \cdot \hat{R}_2)\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)^2 + 2\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)(1 + (\hat{R}_1 \cdot \hat{R}_2)^2) - (\hat{R}_1 \cdot \hat{R}_2) A_2 \\
A_2 &= -\frac{1}{\left(\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right) + (\hat{R}_1 \cdot \hat{R}_2)} \\
B_2 &= \frac{1 - 2(\hat{R}_1 \cdot \hat{R}_2)^2 - \left(-\frac{\hat{R}_2 \cdot \boldsymbol{S}_1}{\hat{R}_1 \cdot \boldsymbol{S}_2}\right)^2}{1 - (\hat{R}_1 \cdot \hat{R}_2)^2} A_2 .
\end{aligned}
\tag{57}
$$

Clearly, the involved appearance of the quantities q^i, p_j of equation (56) depends upon the choice made in equation (51), which in turn depends upon the requirement that the $E(3)$ invariants have the fixed value $\boldsymbol{R} \cdot \boldsymbol{R} = 1$ and $\boldsymbol{R} \cdot \boldsymbol{S} = 0$. Giving up these conditions, allows us to get a simpler expression of the vector $\boldsymbol{R}$ of the global $E(3)$ in terms of the single-particle vectors $\hat{R}_1$ and $\hat{R}_2$ and to thereby achieve a clearer geometric description of the symplectic composition of angular momenta. We put:

$$
\begin{aligned}
&\boldsymbol{R} \longrightarrow \boldsymbol{N} = \tfrac{1}{2}(\hat{R}_1 + \hat{R}_2) \\
&\boldsymbol{\chi} = \hat{R}_1 - \hat{R}_2
\end{aligned}
\tag{58}
$$

so that the invariants of $E(3)$, now enrolled as canonical variables, become:

$$
\begin{aligned}
I_1 &= \boldsymbol{N} \cdot \boldsymbol{N} = \tfrac{1}{2}(1 + \hat{R}_1 \cdot \hat{R}_2) \equiv \mathcal{N}^2 \\
I_2 &= \boldsymbol{N} \cdot \boldsymbol{S} = \tfrac{1}{2}(\hat{R}_1 \cdot \boldsymbol{S}_2 + \hat{R}_2 \cdot \boldsymbol{S}_1).
\end{aligned}
\tag{59}
$$

In this situation, the canonical *completion* of *scheme A* of (35) amounts to building a *scheme B* of the *complete* realization of $E(3)$ (type 2 with $l = k = 2$), so that we have to construct two *supplementary* variables coupled to the invariants. Furthermore, the presence of the 'Abelian' vector $\boldsymbol{N}$ entails that this *complete* realization of $E(3)$ contains two sub-realizations of $SO(3)$ which are just those generated by the 'external' or 'inertial' and 'body' components of the intrinsic angular momentum, respectively (see [25]: equations (39), (41)), (in group-theoretical terms *left* and *right* translations). Actually, in the present case, if $(\boldsymbol{k}_3, \boldsymbol{k}_1, \boldsymbol{k}_2)$ are 'external' unit axes, their 'body' counterparts are $\hat{N} = \frac{\boldsymbol{N}}{\mathcal{N}}$, $\hat{\chi} = \frac{\boldsymbol{\chi}}{\chi} = \frac{(\hat{R}_1 - \hat{R}_2)}{\sqrt{2(1 - \hat{R}_1 \cdot \hat{R}_2)}}$ and $\hat{N} \wedge \hat{\chi}$, respectively. Then the 'body' components of the angular momentum are:

$$
\bar{S}^3 = \boldsymbol{S} \cdot \hat{N} \qquad \bar{S}^1 = \boldsymbol{S} \cdot \hat{\chi} \qquad \bar{S}^2 = \boldsymbol{S} \cdot \hat{N} \wedge \hat{\chi}. \tag{60}
$$

This gives us a hint at the construction of the *supplementary* variable canonically conjugated to $\boldsymbol{S} \cdot \hat{N}$, because the searched variable must formally have the *same geometrical structure* as the variable conjugated to S^3. By canonical completion, we in fact obtain the following two pairs of canonical variables:

$$
\begin{aligned}
&p_1 = \boldsymbol{S} \cdot \hat{N} = \bar{S}^3 = I_2 \qquad p_2 = \sqrt{N \cdot N} = \mathcal{N} = I_1 \\
&q^1 = \tan^{-1} \frac{\boldsymbol{S} \cdot (\hat{N} \wedge \hat{\chi})}{\boldsymbol{S} \cdot \hat{\chi}} = \tan^{-1} \frac{\bar{S}^2}{\bar{S}^1} \qquad q^2 = \frac{(\boldsymbol{S}_1 - \boldsymbol{S}_2) \cdot (\hat{N} \wedge \hat{\chi})}{\chi}.
\end{aligned}
\tag{61}
$$

The difference $\boldsymbol{S}_1 - \boldsymbol{S}_2$ between the particle spin vectors appears in the second *supplementary* variable (q^2 in equation (61)). It is easy to check that, in *geometrical* (not in *group-theoretical* !) terms, q^2 has the same relation to $\mathcal{N}$, as α to $\mathcal{S}$.

In conclusion, the *complete* realization of $E(3)$ can be summarized as follows

$$\begin{array}{|c|c|c|c|} \hline S^3 & \mathcal{S} & \boldsymbol{S}\cdot\hat{N} & \mathcal{N} \\ \tan^{-1}\frac{S^2}{S^1} & \tan^{-1}\frac{\mathcal{S}(\boldsymbol{S}\wedge\hat{N})^3}{[\boldsymbol{S}\wedge(\boldsymbol{S}\wedge\hat{N})]^3} & \tan^{-1}\frac{\boldsymbol{S}\cdot(\hat{N}\wedge\hat{\chi})}{\boldsymbol{S}\cdot\hat{\chi}} & \frac{(\mathcal{S}_1-\mathcal{S}_2)\cdot(\hat{N}\wedge\hat{\chi})}{\chi} \\ \hline \end{array}. \tag{62}$$

The first three canonical pairs contain a simultaneous canonical description of the 'external' and the 'body' intrinsic angular momentum. The *geometric* meaning of the whole set of canonical variables (in particular that of the *supplementary* variables) will be made clear presently. Finally, let us remark in this connection that the geometry of our coordinates should also provide some hints for solving the problem of extracting the physical degrees of freedom of the electromagnetic field via differential operators, exploiting the natural direction $\hat{S}$ instead of the usual radiation condition (see [34]).

The final form of the whole set of canonical variables, which are now adapted both to the constraint structure *and* to the action of the Poincaré group, is summarized in table 5.

7. Relative variables referred to the intrinsic frame of a canonical realization of $E(3)$: geometric meaning of the variables

The geometric interpretation of the canonical variables can be made clear by exploiting the inverse transformation of the one that brings from the *quasi*-SCT to the *typical form*. To this aim, it is instrumental to decompose $\boldsymbol{\pi}_a$ and $\boldsymbol{\rho}_a$ in terms of the vectors $\hat{S}$, $\hat{R}$ and $\hat{S}\wedge\hat{R}$ $(a = 1, \ldots, N-1)$:

$$\begin{cases} \boldsymbol{\rho}_a = A_a \dfrac{\boldsymbol{S}}{\mathcal{S}} + B_a\hat{R} + C_a\dfrac{\boldsymbol{S}}{\mathcal{S}}\wedge\hat{R} \\ \boldsymbol{\pi}_a = D_a \dfrac{\boldsymbol{S}}{\mathcal{S}} + E_a\hat{R} + F_a\dfrac{\boldsymbol{S}}{\mathcal{S}}\wedge\hat{R}. \end{cases} \tag{63}$$

The basis vectors and the coefficients $A_a, B_a, C_a, D_a, E_a, F_a$ have to be expressed as functions of the variables appearing in the *typical form* (see in particular equation (43)).

For simplicity, we shall work out the calculation for the case $N = 3$. First of all, we introduce the vectors (see scheme (62)):

$$\begin{array}{ll} \boldsymbol{N} = \dfrac{\hat{R}_1+\hat{R}_2}{2} & \boldsymbol{\chi} = \dfrac{\hat{R}_1-\hat{R}_2}{2} \\ \boldsymbol{S} = \boldsymbol{S}_1+\boldsymbol{S}_2 & \boldsymbol{W} = \boldsymbol{S}_1-\boldsymbol{S}_2 \end{array} \tag{64}$$

Table 5. System of three particles: chart adapted to the constraint structure *and* to the Poincaré group (p_1 and p_2 are invariants of $E(3)$ in the case of a realization with no fixed invariants).

k^1	k^2	k^3		ϵ	
z^1	z^2	z^3		T	
S^3	$\mathcal{S}$	π_1	π_2	ϵ_1	ϵ_2
$\tan^{-1}\frac{S^2}{S^1}$	α	$\boldsymbol{\rho}_1\cdot\hat{\pi}_1$	$\boldsymbol{\rho}_2\cdot\hat{\pi}_2$	τ_1	τ_2
p_1	p_2				
q^1	q^2				

which satisfy

$$\begin{cases} \boldsymbol{\chi}\cdot\boldsymbol{\chi} = 1 - \boldsymbol{N}\cdot\boldsymbol{N} \\ \boldsymbol{N}\cdot\boldsymbol{\chi} = 0 \\ \boldsymbol{N}\cdot\boldsymbol{S} = -\boldsymbol{\chi}\cdot\boldsymbol{W} \\ \boldsymbol{N}\cdot\boldsymbol{W} = -\boldsymbol{\chi}\cdot\boldsymbol{S} \end{cases} \tag{65}$$

and the Poisson bracket relations†:

$$\begin{aligned} &\{N^i, N^j\} = 0 \\ &\{\chi^i, \chi^j\} = 0 \\ &\{W^i, W^j\} = \epsilon^{ijk} S^k. \end{aligned} \tag{66}$$

Equations (64) provide the relations among $\boldsymbol{N}, \boldsymbol{\chi}, \boldsymbol{S}$ and $\boldsymbol{W}$ and the single-particle vectors $\hat{R}_a$ and $\boldsymbol{S}_a$. Therefore, since equation (44) gives the expression of the single-particle *relative* variables $\boldsymbol{\pi}_a$ and $\boldsymbol{\rho}_a$ as functions of the vectors $\hat{R}_a$ and $\boldsymbol{S}_a$, the inverse transformation can be simply obtained by re-expressing the vectors $\boldsymbol{N}, \boldsymbol{\chi}, \boldsymbol{S}$ and $\boldsymbol{W}$ in terms of the variables of the *typical form*.

The relevance of the geometric decomposition obtained by using the orthonormal frame $\hat{S}$, $\hat{R}$ and $\hat{S}\wedge\hat{R}$ follows from the fact that these vectors are functions of the *irreducible kernel* of the small group of the massive orbits ($E(3)$ has $SO(3)$ as subgroup). Let us put (see *scheme* (62))

$$\begin{cases} \alpha = \tan^{-1} \dfrac{\mathcal{S}(\boldsymbol{S}\wedge\hat{N})^3}{[\boldsymbol{S}\wedge(\boldsymbol{S}\wedge\hat{N})]^3} \\ \beta = \tan^{-1} \dfrac{\boldsymbol{S}\cdot(\hat{N}\wedge\hat{\chi})}{\boldsymbol{S}\cdot\hat{\chi}} \\ \phi = \tan^{-1} \dfrac{S^2}{S^1} \\ \xi = \dfrac{(\boldsymbol{S}_1 - \boldsymbol{S}_2)\cdot(\hat{N}\wedge\hat{\chi})}{\chi} \end{cases} \tag{67}$$

and define the new angle:

$$\psi = \frac{\boldsymbol{N}\cdot\boldsymbol{S}}{\mathcal{N}\mathcal{S}} = \hat{N}\cdot\hat{S}. \tag{68}$$

The geometric setting corresponding to the above definitions is displayed in figures 1 and 2, in which the frame $\hat{S}, \hat{R}, \hat{S}\wedge\hat{R}$ refers to (49) and (52), while the frame $\hat{N}, \hat{\chi}, \hat{N}\wedge\hat{\chi}$ is defined by (58) and subsequent formulae. Finally, we obtain:

$$\begin{aligned} \boldsymbol{N} &= \mathcal{N}\cos\psi\,\hat{S} + \mathcal{N}\sin\psi\,\hat{R} \\ \boldsymbol{\chi} &= (1-\mathcal{N}^2)^{1/2}(\sin\psi\cos\beta\,\hat{S} - \cos\psi\cos\beta\,\hat{R} + \sin\beta\,\hat{S}\wedge\hat{R}) \\ \boldsymbol{W} &= \left(\frac{\mathcal{S}}{\mathcal{N}(1-\mathcal{N}^2)^{1/2}}\sin\psi\cos\psi\cos\beta + \xi\sin\psi\sin\beta\right)\hat{S} \\ &\quad + \left(-\frac{\mathcal{S}}{\mathcal{N}(1-\mathcal{N}^2)^{1/2}}\cos\beta\sin^2\psi + \xi\sin\beta\cos\psi\right)\hat{R} \\ &\quad + \left(\xi\cos\beta + \sin\beta\cos\psi\frac{\mathcal{S}\mathcal{N}}{(1-\mathcal{N}^2)^{1/2}}\right)\hat{S}\wedge\hat{R}. \end{aligned} \tag{69}$$

† Note that these vectors generate the 2-rank groups $[\boldsymbol{N}, \boldsymbol{S}] \equiv E(3)$, $[\boldsymbol{\chi}, \boldsymbol{S}] \equiv E(3)$, $[\boldsymbol{W}, \boldsymbol{S}] \equiv SO(4)$.

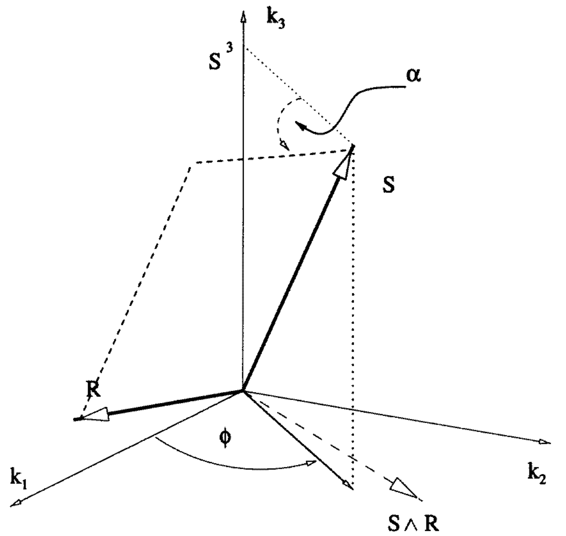

Figure 1. The ϕ and α angles, and the S^3 component. Note that, as regards sections 6 and 7, $\boldsymbol{R}$ has to be replaced by $\boldsymbol{N}$.

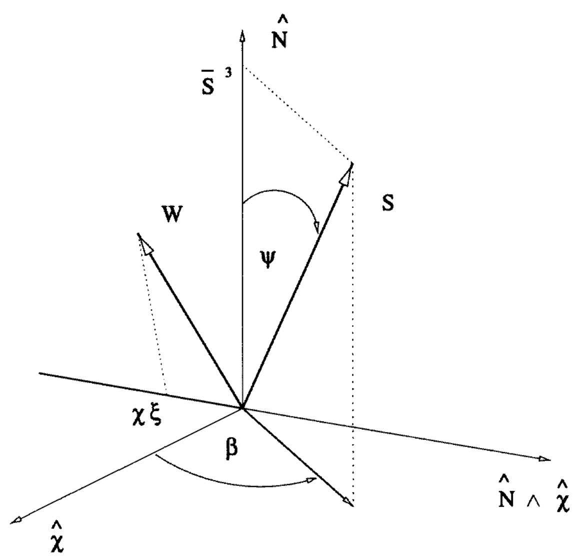

Figure 2. The β and ψ angles, the projection $\chi\xi$ and the $\bar{S}^3$ component.

These expressions, together with equations (63) and (64) provide the desired inverse transformation (see (44) for $N = z$).

8. System of N particles

Finally, the construction of the *typical form* for the system of three particles, gives us a clue for dealing with a system of N particles. We will exploit the 'direct product' of $N - 1$ sub-realizations of $E(3)$ in order to obtain a single realization whose *typical form* has the

Table 6. System of four particles (the first label choice is (ab); p_1 and p_2 are invariants of $E(3)$).

$\boldsymbol{k}$							ϵ		
$\boldsymbol{z}$							T		
$S^3_{(ab)}$	$\mathcal{S}_{(ab)}$	$S^3_{(c)}$	$\mathcal{S}_{(c)}$	$\pi_{(a)}$	$\pi_{(b)}$	$\pi_{(c)}$	ϵ_a	ϵ_b	ϵ_c
$\tan^{-1}\frac{S^2_{(ab)}}{S^1_{(ab)}}$	$\alpha_{(ab)}$	$\tan^{-1}\frac{S^2_{(c)}}{S^1_{(c)}}$	$\alpha_{(c)}$	$\boldsymbol{\rho}_{(a)}\cdot\hat{\boldsymbol{\pi}}_{(a)}$	$\boldsymbol{\rho}_{(b)}\cdot\hat{\boldsymbol{\pi}}_{(b)}$	$\boldsymbol{\rho}_{(c)}\cdot\hat{\boldsymbol{\pi}}_{(c)}$	τ_a	τ_b	τ_c
$p_{1\,(ab)}$	$p_{2\,(ab)}$								
$q^1_{(ab)}$	$q^2_{(ab)}$								

correct number of *inessential* variables.

Since the *irreducible kernel* of any $E(3)$ realization has two 'deegres of freedom', the construction of a single global realization requires the preliminary choice of an initial *label-pair* within the $N-1$ possible ones. Then the *irreducible kernel* of this initial *two-label realization* has to be combined with the kernel of any of the remaining single-particle realizations. The variables of the initial *two-label realization* that do not belong to the *irreducible kernel* become *inessential* variables of a *three-label realization*, which is being constructed in the same way as the *two-label realization* is built up from the single-particle realization. It is now clear how to iterate the procedure until a single global realization of $E(3)$ with a $(N-1)$-label *irreducible kernel* is constructed. Note, finally, that this procedure gives rise to $\binom{N-1}{2}$ different chains of canonical transformations corresponding to $\binom{N-1}{2}$ final realizations of $E(3)$.

We shall carry out explicitly the construction of a *typical form* for a system of *four* particles. We begin as usual with table 1 and rewrite the variables of the *irreducible kernel* in the 'direct product' form†.

$S^3_{(a)}$	$\mathcal{S}_{(a)}$	$S^3_{(b)}$	$\mathcal{S}_{(b)}$	$S^3_{(c)}$	$\mathcal{S}_{(c)}$
$\tan^{-1}\frac{S^2_{(a)}}{S^1_{(a)}}$	$\alpha_{(a)}$	$\tan^{-1}\frac{S^2_{(b)}}{S^1_{(b)}}$	$\alpha_{(b)}$	$\tan^{-1}\frac{S^2_{(c)}}{S^1_{(c)}}$	$\alpha_{(c)}$

The explicit procedure is as follows. First, two labels are selected and the first *irreducible kernel* is constructed (see table 6).

$S^3_{(ab)}$	$\mathcal{S}_{(ab)}$
$\tan^{-1}\frac{S^2_{(ab)}}{S^1_{(ab)}}$	α

The form of α obviously depends on the choice of the vector $\hat{R}_{(ab)}$ of equation (43), which now satisfies conditions (51).

Then, the *irreducible kernel* of the *three-label realization* is constructed by defining the two vectors‡ (see equation 58)

$$\boldsymbol{N}_{((ab)c)} = \tfrac{1}{2}(\hat{R}_{(ab)} + \hat{R}_{(c)}) \tag{70}$$

and

$$\boldsymbol{\chi}_{((ab)c)} = \tfrac{1}{2}(\hat{R}_{(ab)} - \hat{R}_{(c)}). \tag{71}$$

This leads from table 7 to table 8. The angle α of the *irreducible kernel* is now:

$$\alpha_{((ab)c)} = \tan^{-1}\frac{\mathcal{S}[\boldsymbol{S}\wedge\boldsymbol{N}_{((ab)c)}]^3}{[\boldsymbol{S}\wedge[\boldsymbol{S}\wedge\boldsymbol{N}_{((ab)c)}]]^3}. \tag{72}$$

† The *inessential* variables of the single-particle realizations are not listed for simplicity.

‡ Again, the remaining degrees of freedom are realized by *inessential* variables.

Table 7. System of four particles: chart adapted to the constraint structure (first label choice ((ab)c)).

$\boldsymbol{k}$ $\boldsymbol{z}$					ϵ T		
S^3 $\tan^{-1}\frac{S^2}{S^1}$	$\mathcal{S}$ $\alpha_{((ab)c)}$	$\pi_{(a)}$ $\boldsymbol{\rho}_{(a)}\cdot\hat{\boldsymbol{\pi}}_{(a)}$	$\pi_{(b)}$ $\boldsymbol{\rho}_{(b)}\cdot\hat{\boldsymbol{\pi}}_{(b)}$	$\pi_{(c)}$ $\boldsymbol{\rho}_{(c)}\cdot\hat{\boldsymbol{\pi}}_{(c)}$	ϵ_1 τ_1	ϵ_2 τ_2	ϵ_3 τ_3
$p_{1\,(ab)}$ $q^1_{(ab)}$	$p_{2\,(ab)}$ $q^2_{(ab)}$	$p_{1((ab)c)}$ $q^1_{((ab)c)}$	$p_{2\,((ab)c)}$ $q^2_{((ab)c)}$				

Table 8. System of four particles: chart adapted to the constraint structure *and* to the Poincaré group (first label choice ((ab)c)).

$\boldsymbol{k}$ $\boldsymbol{z}$		
S^3 $\tan^{-1}\frac{S^2}{S^1}$	$\mathcal{S}$ $\alpha_{((ab)c)}$	
$\pi_{(a)}$ $\boldsymbol{\rho}_{(a)}\cdot\hat{\boldsymbol{\pi}}_{(a)}$	$\pi_{(b)}$ $\boldsymbol{\rho}_{(b)}\cdot\hat{\boldsymbol{\pi}}_{(b)}$	$\pi_{(c)}$ $\boldsymbol{\rho}_{(c)}\cdot\hat{\boldsymbol{\pi}}_{(c)}$
$\boldsymbol{S}_{(ab)}\cdot\hat{N}_{(ab)}$ $\tan^{-1}\frac{\boldsymbol{S}_{(ab)}\cdot\hat{\chi}_{(ab)}}{\boldsymbol{S}_{(ab)}\cdot(\hat{N}_{(ab)}\wedge\hat{\chi}_{(ab)})}$	$\mathcal{N}_{(ab)}$ $\frac{(\boldsymbol{S}_{(a)}-\boldsymbol{S}_{(b)})\cdot(\hat{N}_{(ab)}\wedge\hat{\chi}_{(ab)})}{\chi_{(ab)}}$	
$\boldsymbol{S}\cdot\hat{N}_{((ab)c)}$ $\tan^{-1}\frac{\boldsymbol{S}\cdot\hat{\chi}_{((ab)c)}}{\boldsymbol{S}\cdot\hat{N}_{((ab)c)}\wedge\hat{\chi}_{((ab)c)}}$	$\mathcal{N}_{((ab)c)}$ $\frac{(\boldsymbol{S}_{(ab)}-\boldsymbol{S}_{(c)})\cdot\hat{N}_{((ab)c)}\wedge\hat{\chi}_{((ab)c)}}{\chi_{((ab)c)}}$	

The *typical form* clearly depends on the choice of the chain of labels (see table 8). Corresponding to each choice of labels, a basic reference frame in the $\Re^3$ space of *Wigner vectors* is defined. The vector $\boldsymbol{S}$ is shared by any of the *typical forms* corresponding to any possible different chain, while the unit vector $\hat{R}$ depends on any given chain according to equation 43), with α given by equation (72). The $\binom{4-1}{2}$ possible reference frames differ from one another by a spatial rotation of the vectors $\boldsymbol{R}$ e $\boldsymbol{S}\wedge\boldsymbol{R}$ around the $\boldsymbol{S}$ direction. The rotation angle relating two *typical forms* is simply given by the difference between the two variables conjugated to $\mathcal{S}$. The geometry is as follows: if $\boldsymbol{a}$ and $\boldsymbol{b}$ (like $\hat{R}_a$ and $\hat{R}_b$) are two vectors with $\{a^i, a^j\} = \{b^i, b^j\} = 0$, from the expressions

$$\begin{aligned}\alpha_a &= \tan^{-1} \frac{\mathcal{S}(\boldsymbol{S} \wedge \boldsymbol{a})^3}{[\boldsymbol{S} \wedge (\boldsymbol{S} \wedge \boldsymbol{a})]^3} \\ \alpha_b &= \tan^{-1} \frac{\mathcal{S}(\boldsymbol{S} \wedge \boldsymbol{b})^3}{[\boldsymbol{S} \wedge (\boldsymbol{S} \wedge \boldsymbol{b})]^3}\end{aligned} \tag{73}$$

we obtain the angle

$$\alpha_a - \alpha_b = \tan^{-1} \frac{\boldsymbol{a} \cdot (\boldsymbol{S} \wedge \boldsymbol{b})}{\boldsymbol{b} \cdot (\boldsymbol{S} \wedge (\boldsymbol{S} \wedge \boldsymbol{b}))}. \tag{74}$$

Of course, equation (74) does not exhaust the description of the difference between any pair of *typical forms*: the transformation between the *inessential variables* is needed too.

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